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STOCHASTIC PERTURBATION APPROACH TO ENGINEERING STRUCTURE VIBRATIONS BY THE FINITE DIFFERENCE METHOD

M. Kamiński†

Department of Civil Engineering 215, *George Brown School of Engineering*, *Rice University*, 6100 *Main Street*, *Houston*, 77005-1892 *TX*, *U.S.A. E-mail*: *kaminski@ruf.rice.edu*

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The main idea of the paper is to introduce the second order perturbation second probabilistic moment analysis in the context of the finite difference method (FDM) modelling of vibrations. The approach can be successfully applied in all those engineering analyses where FDM modelling of engineering structures vibrations is still useful and, at the same time, some structural parameters are random variables or fields. The general advantage of the stochastic finite difference method (SFDM) proposed is the relatively easy extension of the existing deterministic results of the classical elastodynamics on the random or stochastic case. However, similarly to stochastic boundary or finite element methods, the approach proposed has its limitations on the second order random uncertainties measures of input random variables.

1. INTRODUCTION

The finite difference method (FDM) [1–3] being probably one of the oldest numerical techniques in engineering and scientific computations is still widely used in the domain of heat transfer [4], electrodynamics [5], electromagnetics [6] and in hydromechanics [7] even for large-scale problems [8]. On the other hand, engineers and scientists usually obtain the information about structural and system parameters in the form of statistical estimators or random processes. Because of that, the idea of extension of the FDM on the random variables, fields or processes modelling seems to be quite natural, and promising.

Due to such an extension, one can answer the question how the probabilistic parameters of input result in corresponding parameters of the structural response. It is known that it can be done using various mathematical and numerical methods [9–11]. The second order perturbation second central probabilistic moments technique [10, 13] can be applied, for instance, to show how to build up the stochastic solution starting from the corresponding deterministic one. The alternative ways of random processes modelling to the method proposed, in the context of a FDM stochastic extension, is the application of the Monte-Carlo simulation technique [14] or stochastic spectral techniques (improved Neumann, Karhunen–Loeve or Polynomial Chaos expansions [9]).

The stochastic perturbation technique is used here, analogously to its applications in the finite element method [10] as well as the boundary element method [12] stochastic

†Permanent address: Division of Mechanics of Materials, Technical University of Lodz Al. Politechniki 6, 93-590 Lodz, Poland. E-mail: marcinka@ck-sg.p.lodz.pl

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extensions, to utilize the FDM for the needs of probabilistic analysis [13]. Considering the fact that input statistical information on random variables in engineering problems is usually given using up to the second moment statistics-expected values and standard deviations (variances or cross-correlations), then the second order perturbation method is proposed. The application of higher than the second order perturbation methods needs reliable data on higher order probabilistic moments that is not always available. Furthermore, in numerous applications, the third order probabilistic coefficients are almost equal to 0, while application of the fourth order perturbation analysis complicates decisively the equilibrium equations solution. Finally, let us observe that most advanced tools for reliability analysis are still based on the second order probabilistic analysis (second order reliability methods (SORM)); so, from this particular point of view, there is no need to carry out higher order perturbation analysis.

Considering the above, the application of the second order perturbation technique is shown for the analysis of vibrations of structures with random parameters which is still of interest [9, 11, 15]. The proposed approach is illustrated by the example of free vibrations of a fixed–pinned beam and simply supported square plate. The results obtained in the paper in the context of the application of the stochastic perturbation-based finite difference methodology to vibration analysis [2] can be used to extend any existing FDM-based computer programs to analyze problems with one or more parameters being random variables or processes. In further extensions of the computational techniques for applications in probabilistic mechanics and engineering, the usage of the FDM is recommended considering the fact that this method is efficiently used for time discretization of the analyzed structures as well. Because of that and considering the numerical modelling of stochastic stationary and unstationary processes, the application of the specially adopted FDM should be taken into account.

2. STOCHASTIC SECOND ORDER PERTURBATION APPROACH

Generally, we are looking for the solution of the problem

$$M\ddot{u} + C\dot{u} + Ku = f, \tag{1}$$

where M, C and K are linear stochastic operators, u is the random structural response, while f is the admissible excitation of the system. Usually, such operators are identified as mass, damping and stiffness matrices in structural dynamics applications. As it is known [16], the analytical solutions for such a class of partial differential equations are available for some specific cases and that is why quite different approximating numerical methods are used. Various mathematical approaches to the solution of that problem are reported and presented in references [9, 17], however the second order perturbation second central probabilistic moment approach is proposed below.

For this purpose, let us denote the random variable of the problem as a vector $\{b^r(\mathbf{x};\theta)\}$ and its probability density as $p(b^r)$ and $p(b^r, b^s)$, respectively, where $r,\ s = 1,\ 2, \dots, R$ index the random variables. Next, the expected values of this vector are defined as

$$E[b^r] = \int_{-\infty}^{+\infty} b^r p(b^r)\,\mathrm{d}b^r \tag{2}$$

and the covariance in the following way!

$$\mathrm{Cov}(b^r, b^s) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (b^r - E[b^r])(b^s - E[b^s])\,p(b^r, b^s)\,\mathrm{d}b^r\,\mathrm{d}b^s. \tag{3}$$

If the discrete representation of random variable $b(\mathbf{x};\theta)$ is used, the statistical estimators may be applied to approximate any order probabilistic moments of this variable.

Next, all material and physical parameters of Ω as well as the state functions, being random fields, are extended by the use of the stochastic expansion via the Taylor series as follows [10]:

$$F(b) = F^0(b^0) + \sum_{n=1}^{N}\left(\frac{\varepsilon^n}{n!}F^{(n)}(b)\prod_{r=1}^{N}\Delta b^r\right), \tag{4}$$

where ε is a given small perturbation, $\varepsilon\Delta b^r$ denotes the first order variation of Δb^r about its expected value $E[b^r]$ and $F^{(n)}(b)$ represents the n-th order partial derivatives with respect to the random variables, while $F^{(n)}(b^0)$ denotes the n-th order partial derivative calculated at the expected value of the input random variables vector. Variable θ represents the random event belonging to the additional probabilistic space of admissible events (non-negative, for instance). The second order perturbation approach is introduced here and hence, the analyzed random function $F(b)$ is extended as

$$F(b) = F^0(b^0) + \varepsilon F^{,r}(b)\Delta b^r + \frac{\varepsilon^2}{2}F^{,rs}(b)\Delta b^r\Delta b^s. \tag{5}$$

Let us note that the second order equation is obtained by multiplying the R-variate probability density function $p_R(b_1, b_2, \ldots, b_R)$ by the ε^2-terms and integrating over the domain of the random field variables $\mathbf{b}(\mathbf{x};\theta)$. Then, the L.H.S. differential operator $K(b)$ introduced in equation (1) is extended as

$$K(b) = K^0(b^0) + K^{,r}(b)\Delta b^r + K^{,rs}(b)\Delta b^r\Delta b^s \tag{6}$$

under the assumption that the small parameter of the extension ε is equal to 1. Further, applying the stochastic second order Taylor series based extension to the basic deterministic equation (1) of the problem and equating the same order terms, the following relations are obtained for $\tau\in[0,\ \infty)$:

Zeroth order equations:

$$M^0(b^0)\ddot{u}^0(b^0;\tau) + C^0(b^0)\dot{u}^0(b^0;\tau) + K^0(b^0)u^0(b^0;\tau) = f^0(b^0;\tau). \tag{7}$$

First order equations (*for* $r = 1, \ldots, R$):

$$\begin{aligned} &M^{,r}(b^0)\ddot{u}^0(b^0;\tau) + C^{,r}(b^0)\dot{u}^0(b^0;\tau) + K^{,r}(b^0)u^0(b^0;\tau)\\ &\quad + M^0(b^0)\ddot{u}^{,r}(b^0;\tau) + C^0(b^0)\dot{u}^{,r}(b^0;\tau) + K^0(b^0)u^{,r}(b^0;\tau) = f^{,r}(b^0;\tau). \end{aligned} \tag{8}$$

Second order equations (*for* $r, s = 1, \ldots, R$):

$$\begin{aligned} &M^{,rs}(b^0)\ddot{u}^0(b^0;\tau) + C^{,rs}(b^0)\dot{u}^0(b^0;\tau) + K^{,rs}(b^0)u^0(b^0;\tau)\\ &\quad + 2M^{,r}(b^0)\ddot{u}^{,s}(b^0;\tau) + 2C^{,r}(b^0)\dot{u}^{,s}(b^0;\tau) + 2K^{,r}(b^0)u^{,s}(b^0;\tau)\\ &\quad + M^0(b^0)\ddot{u}^{,rs}(b^0;\tau) + C^0(b^0)\dot{u}^{,rs}(b^0;\tau) + K^0(b^0)u^{,rs}(b^0;\tau) = f^{,rs}(b^0;\tau). \end{aligned} \tag{9}$$

If we are looking for the n-th order partial differential perturbation-based equation, it can be rewritten generalizing the statement posed above in the form

$$\sum_{k=0}^{n}\binom{n}{k}M^{(n)}(b^0)\ddot{u}^0(b^0;\tau)+\sum_{k=0}^{n}\binom{n}{k}C^{(n)}(b^0)\dot{u}^0(b^0;\tau)+\sum_{k=0}^{n}\binom{n}{k}K^{(n)}(b^0)u^0(b^0;\tau)=f^{(n)}(b^0;\tau). \tag{10}$$

Assuming that higher than the second order perturbations can be neglected, this equations system constitutes our equilibrium problem. The detailed convergence studies should be carried out in further extensions of the model with respect to the perturbation order, parameter θ and coefficient of variation of input random variables.

Further, it is observed that equations system (8) is rewritten for all random parameters of the problem indexed by $r = 1, \dots, R$ (R equations), while system (9) gives us generally R^2 equations. To eliminate the large number of equations we multiply equation (9) by the covariance matrix of input random parameters and therefore

Zeroth order equations:

$$M^0(b^0)\ddot{u}^0(b^0;\tau)+C^0(b^0)\dot{u}^0(b^0;\tau)+K^0(b^0)u^0(b^0;\tau)=f^0(b^0;\tau). \tag{11}$$

First order equations (*for* $r = 1, \dots, R$):

$$\begin{aligned}M^0(b^0)\ddot{u}^{,r}(b^0;\tau)+C^0(b^0)\dot{u}^{,r}(b^0;\tau)+K^0(b^0)u^{,r}(b^0;\tau)&=f^{,r}(b^0;\tau)\\ -[M^{,r}(b^0)\ddot{u}^0(b^0;\tau)+C^{,r}(b^0)\dot{u}^0(b^0;\tau)&+K^{,r}(b^0)u^0(b^0;\tau)\}.\end{aligned} \tag{12}$$

Second order equations (*for* $r, s = 1, \dots, R$):

$$\begin{aligned}&M^0(b^0)\ddot{u}^{(2)}(b^0;\tau)+C^0(b^0)\dot{u}^{(2)}(b^0;\tau)+K^0(b^0)u^{(2)}(b^0;\tau)\\ &=\{f^{,rs}(b^0;\tau)-M^{,rs}(b^0)\ddot{u}^0(b^0;\tau)-C^{,rs}(b^0)\dot{u}^0(b^0;\tau)-K^{,rs}(b^0)u^0(b^0;\tau)\\ &-2M^{,r}(b^0)\ddot{u}^{,s}(b^0;\tau)-2C^{,r}(b^0)\dot{u}^{,s}(b^0;\tau)-2K^{,r}(b^0)u^{,s}(b^0;\tau)\}\,\mathrm{Cov}(b^r,b^s).\end{aligned} \tag{13}$$

Let us observe that for the n-th order perturbation approach, the closure of hierarchical equations is obtained by the n-th order correlation of input random process components b^r and b^s, respectively. For this purpose, n-th order statistical information about input random variables is, however, necessary.

To obtain the probabilistic solution for the considered equilibrium problem, equation (11) is solved for u^0 (and its time derivatives $\dot{u}^0$ and $\ddot{u}^0$, correspondingly), followed by equation (12) solved for the first order terms of $u^{,r}$ and, finally, equation (13), for $u^{(2)}$. The two probabilistic moment characterization of all the state functions starts from the expected value of the displacements. Using its definition

$$E[u(t)]=\int_{-\infty}^{+\infty}u(t)p_R(\mathbf{b}(\mathbf{x};\theta))\,\mathrm{d}\mathbf{b}, \tag{14}$$

the second order accurate expectations are equal to

$$E[u(t)]=u^0(t)+\tfrac{1}{2}u^{,rs}S_b^{rs}=u^0(t)+\tfrac{1}{2}u^{(2)}. \tag{15}$$

In quite a similar manner, we obtain second order probabilistic characteristics. Defining covariance as

$$\mathrm{Cov}(u(t_1); u(t_2)) = \int_{-\infty}^{+\infty} \{u(t_1) - E[u(t_1)]\}\{u(t_2) - E[u(t_2)]\} p_R(\mathbf{b}(\mathbf{x};\theta))\,\mathrm{d}\mathbf{b}, \tag{16}$$

it is obtained that

$$\mathrm{Cov}(u(t_1); u(t_2)) = u^{,r}(t_1) u^{,s}(t_2)\,\mathrm{Cov}(b^r, b^s). \tag{17}$$

This completes the two-moment characterization of the perturbation-based solution for dynamic equilibrium problem (1). The entire solution simplifies in the case of free vibrations when the following equation is to be solved:

$$[K - \Omega_{(\alpha)} M]\Phi = 0, \tag{18}$$

where $\Omega_{(\alpha)}$ and Φ are the eigenvalues and eigenvectors, respectively, and $\alpha = 1, \ldots, N$ represents the total number of degrees of freedom of a structure. Using the second order expansion, we obtain the following equations system:

$$[K^0 - \Omega^0_{(\alpha)} M^0]\Phi^0 = 0, \tag{19}$$

$$[K^0 - \Omega^0_{(\alpha)} M^0]\Phi^{,r} = -[K^{,r} - \Omega^{;r}_{(\alpha)} M^0 - \Omega^0_{(\alpha)} M^{,r}]\Phi^0, \tag{20}$$

$$[K^0 - \Omega^0_{(\alpha)} M^0]\Phi^{(2)} = -\{[K^{,rs} - \Omega^{;rs}_{(\alpha)} M^0 - 2\Omega^{;r}_{(\alpha)} M^{,s} - \Omega^0_{(\alpha)} M^{,rs}]\Phi^0$$

$$+ 2[K^{,r} - \Omega^{;r}_{(\alpha)} M^0 - \Omega^0_{(\alpha)} M^{,r}]\Phi^{,s}\}\,\mathrm{Cov}(b^r, b^s). \tag{21}$$

To determine probabilistic moments of the eigenvectors, up to the second order derivatives with respect to input random variables are to be determined. While zeroth order quantities are obtained from relation (19) directly, the methodology to calculate first order terms is definitely more complicated. Equation (20) is transformed for this purpose by multiplying by the transposed zeroth order eigenvector and it is obtained that

$$\Phi^{0^{\mathrm{T}}}[K^0 - \Omega^0_{(\alpha)} M^0]\Phi^{,r} - \Phi^{0^{\mathrm{T}}}\Omega^{;r}_{(\alpha)} M^0 \Phi^0 = -\Phi^{0^{\mathrm{T}}}[K^{,r} - \Omega^0_{(\alpha)} M^{,r}]\Phi^0. \tag{22}$$

Since Φ^0 is diagonal and K^0 and M^0 are symmetric, equation (22) is obtained as

$$[\Phi^{0^{\mathrm{T}}}[K^0 - \Omega^0_{(\alpha)} M^0]\Phi^{,r}]^{\mathrm{T}} = \Phi^{,r^{\mathrm{T}}}[K^0 - \Omega^0_{(\alpha)} M^0]\Phi^0 = 0. \tag{23}$$

Let us observe that $\Omega^{,r}$ is diagonal and therefore

$$\Phi^{0^{\mathrm{T}}}\Omega^{;r}_{(\alpha)} M^0 \Phi^0 = \Omega^{;r}_{(\alpha)}\Phi^{0^{\mathrm{T}}} M^0 \Phi^0, \tag{24}$$

which finally gives

$$\Omega^{;r}_{(\alpha)} = \Phi^{0^{\mathrm{T}}}[K^{,r} - \Omega^0_{(\alpha)} M^{,r}]\Phi^0. \tag{25}$$

Next, using an analogous technique in case of the second order equation, it is derived that

$$\Phi^{0^{\mathrm{T}}}[K^0 - \Omega^0_{(\alpha)}M^0]\Phi^{(2)} - \Phi^{0^{\mathrm{T}}}\Omega^{(2)}_{(\alpha)}M^0\Phi^0$$

$$= -\{\Phi^{0^{\mathrm{T}}}[K^{,rs} - 2\Omega^{;r}_{(\alpha)}M^{,s} - \Omega^0_{(\alpha)}M^{,rs}]\Phi^0 + 2\Phi^{0^{\mathrm{T}}}[K^{,r} - \Omega^{;r}_{(\alpha)}M^0 - \Omega^0_{(\alpha)}M^{,r}]\Phi^{,s}\}\,\mathrm{Cov}(b^r, b^s), \tag{26}$$

which finally implies that

$$\Phi^{(2)} = \{\Phi^{0^{\mathrm{T}}}[K^{,rs} - 2\Omega^{;r}_{(\alpha)}M^{,s} - \Omega^0_{(\alpha)}M^{,rs}]\Phi^0 + 2\Phi^{0^{\mathrm{T}}}[K^{,r} - \Omega^{;r}_{(\alpha)}M^0 - \Omega^0_{(\alpha)}M^{,r}]\Phi^{,s}\}\,\mathrm{Cov}(b^r, b^s). \tag{27}$$

The next problem is to determine the first and second order derivatives of the eigenvectors. Basically, the eigenvector derivative is expressed as a linear combination of all the eigenvectors in the original system. Equations describing the coefficients of the linear combination are composed using the M-orthonormality and K-orthogonality conditions [9]. Starting from equation (20), the α-th eigenpair is rewritten as

$$[K^0 - \Omega^0_{(\alpha)}M^0]\Phi^{;r}_{(\alpha)} = -[K^{,r} - \Omega^{;r}_{(\alpha)}M^0 - \Omega^0_{(\alpha)}M^{,r}]\Phi^0_{(\alpha)} \tag{28}$$

and equation (25) in the following form:

$$\Omega^{;r}_{(\alpha)} = \Phi^0_{(\alpha)}[K^{,r} - \omega^0_{(\alpha)}M^{,r}]\Phi^0_{(\alpha)}. \tag{29}$$

By substitution, these holds

$$[K^0 - \Omega^0_{(\alpha)}M^0]\Phi^{;r}_{(\alpha)} = R^r_{(\alpha)} \tag{30}$$

with $R^r_{(\alpha)}$ being equal to

$$R^r_{(\alpha)} = -[K^{,r} - \Phi^0_{(\alpha)}(K^{,r} - \Omega^0_{(\alpha)}M^{,r})\Phi^0_{(\alpha)}M^0 - \Omega^0_{(\alpha)}M^{,r}]\Phi^0_{(\alpha)}. \tag{31}$$

Further, we assume that the α-th first order eigenvector $\Phi^{;r}_{(\alpha)}$ can be expressed as a linear combination of all zeroth order eigenvectors as

$$\Phi^{;r}_{(\alpha)} = \Phi^0 a^r_{(\alpha)}. \tag{32}$$

The complete description of the coefficients $a^r_{(\alpha)}$ is given by [10]

$$a^r_{(\alpha)} = \begin{cases} \dfrac{\Phi^0_{(\alpha)}R^r_{(\hat{\alpha})}}{\Omega^0_{(\alpha)} - \Omega^0_{(\hat{\alpha})}} & \text{for } \alpha \neq \hat{\alpha}, \\ -\frac{1}{2}\Phi^0_{(\hat{\alpha})}M^{,r}\Phi^0_{(\hat{\alpha})} & \text{for } \alpha = \hat{\alpha}. \end{cases} \tag{33}$$

Similarly as above, the second order eigenvector $\Phi^{(2)}_{(\alpha)}$ is approximated by a linear combination of all the zeroth order eigenvectors by

$$\Phi^{(2)}_{(\alpha)} = \Phi^0 a^{(2)}_{(\alpha)}. \tag{34}$$

Then, one can demonstrate the following result [10]:

$$a^{(2)}_{(\hat{\alpha})} = \begin{cases} \dfrac{\Phi^0_{(\alpha)} R^{(2)}_{(\hat{\alpha})}}{\Omega^0_{(\alpha)} - \Omega^0_{(\hat{\alpha})}} \quad \text{for } \alpha \neq \hat{\alpha}, \\ -\left(\frac{1}{2}\Phi^0_{(\hat{\alpha})} M^{,rs} \Phi^0_{(\hat{\alpha})} + 2\Phi^0_{(\hat{\alpha})} M^{,r} \Phi^{,s}_{(\hat{\alpha})} + a^r_{(\hat{\alpha})} a^s_{(\hat{\alpha})}\right) \mathrm{Cov}(b^r, b^s) \quad \text{for } \alpha = \hat{\alpha}. \end{cases} \tag{35}$$

Finally, the first two probabilistic moments of the eigenvalues and eigenvectors are found in a way which completes the solution of the second order second central probabilistic moment eigenvalue and eigenvector problem.

3. VIBRATIONS ANALYSIS OF SOME ENGINEERING STRUCTURES

The general methodology presented in the previous section is applied to the stochastic analysis of vibrations and illustrated with examples of a Euler–Bernoulli beam as well as a rectangular square plate supported at its corners. The obtained results show that the application of the second order perturbation technique gives satisfactory results in comparison with the exact probabilistic solution obtained using the definition of the expected values and variances.

3.1. VIBRATION OF EULER–BERNOULLI BEAM

As it is known, the equation of motion for the forced lateral vibration of a non-uniform beam has the form

$$\frac{\partial^2}{\partial x^2}\left[eI(x) \frac{\partial^2 w}{\partial x^2}(x, t) \right] + \rho A(x) \frac{\partial^2 w}{\partial t^2}(x, t) = f(x, t), \tag{36}$$

where $f(x, t)$ is the external force per unit length of the beam, ρ is the mass density of the beam, $A(x)$ is its the cross-sectional area. Assuming constant cross-section of the beam along its length, i.e., $eI(x) = eI = const.$ and $A(x) = A = const.$, equation (36) can be rewritten as

$$eI \frac{\partial^4 w}{\partial x^4}(x, t) + \rho A \frac{\partial^2 w}{\partial t^2}(x, t) = f(x, t), \tag{37}$$

while for the free vibrations it is obtained that

$$c^2 \frac{\partial^4 w}{\partial x^4}(x, t) + \frac{\partial^2 w}{\partial t^2}(x, t) = 0, \tag{38}$$

where

$$c = \sqrt{\frac{eI}{\rho A}}. \tag{39}$$

After the separation of space and time variables,

$$w(x, t) = W(x) T(t), \tag{40}$$

the following two equations are obtained for free vibrations:

$$\frac{\partial^4 W(x)}{\partial x^4} - \beta^4 W(x) = 0, \qquad \frac{\partial^2 T(t)}{\partial t^2} + \omega^2 T(t) = 0, \tag{41}$$

where

$$\beta^4 = \frac{\omega^2}{c^2} = \frac{\rho A \omega^2}{eI}. \tag{42}$$

The characteristic function of the beam can be obtained as

$$W(x) = C_1 \exp(\beta x) + C_2 \exp(-\beta x) + C_3 \exp(\mathrm{i}\beta x) + C_4 \exp(-\mathrm{i}\beta x), \tag{43}$$

while ω being the natural frequency of vibration can be expressed as

$$\omega = \beta^2 \sqrt{\frac{eI}{\rho A}}. \tag{44}$$

Applying the stochastic second order perturbation methodology, equation (41) can be rewritten in the form of up to the second order equations system as follows:

Zeroth order equation:

$$W^0_{,xxxx}(x) - (\beta^4)^0 W^0(x) = 0. \tag{45}$$

First order (*R simultaneous*) *equations*:

$$W^{;r}_{,xxxx}(x) - ((\beta^4)^{,r} W^0(x) + (\beta^4)^0 W^{,r}(x)) = 0,\ r,\ s = 1, \ldots, R. \tag{46}$$

Second order (R*($R + 1$)) *equations*:

$$W^{;rs}_{,xxxx}(x) - ((\beta^4)^{,rs} W^0(x) + 2(\beta^4)^{,r} W^{,s}(x) + (\beta^4)^0 W^{,rs}(x)) = 0,\ r,\ s = 1, \ldots, R. \tag{47}$$

The β^4 are parameters of probabilistic derivatives that can be calculated directly. Finally, to obtain the single second order equation, the last formula is written in the form

$$W^{(2)}_{,xxxx}(x) - ((\beta^4)^{,rs} W^0(x) + 2(\beta^4)^{,r} W^{,s}(x) + (\beta^4)^0 W^{,rs}(x))\,\mathrm{Cov}(b^r, b^s) = 0, \tag{48}$$

where

$$W^{(2)}_{,xxxx}(x) = W^{;rs}_{,xxxx}(x)\,\mathrm{Cov}(b^r, b^s). \tag{49}$$

The relevant orders of the function $W(x)$ are calculated as

$$W_{(n)}(x) = C_1 \cos^{(n)} \beta x + C_2 \sin^{(n)} \beta x + C_3 \cosh^{(n)} \beta x + C_4 \sinh^{(n)} \beta x, \tag{50}$$

where the following notation is used:

$$\cos^{(n)} \beta x = \frac{\partial^n}{\partial b^n} (\cos \beta x), \text{ etc.,} \tag{51}$$

while the constants $C_j,\ j=1,\dots,4$ are calculated from the well-known boundary conditions. Finally, the expected values of the characteristic function are calculated as

$$E[W(x)] = W^0(x) + \tfrac{1}{2}W^{,rs}(x)\,\mathrm{Cov}(b^r, b^s) \tag{52}$$

and cross-covariances as

$$\mathrm{Cov}_w(x^{(1)}; x^{(2)}) = W^{,r}(x^{(1)})\,W^{,s}(x^{(2)})\,\mathrm{Cov}(b^r, b^s). \tag{53}$$

This completes the second order perturbation second central probabilistic moment approach to the vibration analysis of beam structures.

3.2. ANALYSIS OF RECTANGULAR PLATES

Starting from Love's equations [18] for shell equilibrium and applying zero curvatures, the resulting expression can be rewritten as

$$D\nabla^4 w + \rho h\ddot{w} = q, \tag{54}$$

where D denotes the plate stiffness

$$D = \frac{eh^3}{12(1-\nu^2)}, \tag{55}$$

while ρ, e, ν, h and q denote the mass density, Young's modulus, the Poisson ratio and the thickness of the plate as well as the magnitude of a uniformly distributed load applied on the structure.

Relation (54) is usually rewritten in rectangular co-ordinates as

$$D(w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}) + \rho h\ddot{w} = q. \tag{56}$$

Looking for natural frequencies, the following transformation is applied:

$$w = W\exp(\mathrm{j}\omega t), \tag{57}$$

which gives

$$D(W_{,xxxx} + 2W_{,xxyy} + W_{,yyyy}) + \rho h\omega^2 W = 0. \tag{58}$$

The solution can be carried further only if detailed boundary conditions are specified. If, for instance, the plate has dimensions $a\times b$ corresponding to the x- and y-axis and is simply supported along the axes parallel to the y direction, the solution can be expressed as

$$W(x, y) = Y(y)\sin\frac{m\pi x}{a}, \tag{59}$$

which gives the motion equation in the form

$$Y_{,yyyy} - 2\left(\frac{m\pi}{a}\right)^2 Y_{,yy} + \left[\left(\frac{m\pi}{a}\right)^4 - \frac{\rho h}{D}\,\omega^2\right] Y = 0, \tag{60}$$

where

$$Y(y) = \sum_{i=1}^{4} C_i \exp(\lambda_i (y/b)). \tag{61}$$

Further, it is obtained that

$$\lambda_i = \pm \frac{b}{a} m\pi \sqrt{1 \pm K} \tag{62}$$

and

$$K = \frac{\omega}{(m^2\pi^2/a^2)\sqrt{D/(\rho h)}}. \tag{63}$$

Application of the second order stochastic perturbation technique makes it possible to rewrite equation (56), in terms of input random field $b^r(x)$, $r = 1, \ldots, R$, as

Zeroth order relation:

$$D^0(W^0_{,xxxx} + 2W^0_{,xxyy} + W^0_{,yyyy}) = (\rho h \omega^2)^0 W^0. \tag{64}$$

First order relations ($r = 1, \ldots, R$):

$$D^0(W^{,r}_{,xxxx} + 2W^{,r}_{,xxyy} + W^{,r}_{,yyyy}) = (\rho h\omega^2)^{,r} W^0 + (\rho h \omega^2)^0 W^{,r} - D^{,r}(W^0_{,xxxx} + 2W^0_{,xxyy} + W^0_{,yyyy}). \tag{65}$$

Second order relation:

$$D^0(W^{(2)}_{,xxxx} + 2W^{(2)}_{,xxyy} + W^{(2)}_{,yyyy}) = \{(\rho h\omega^2)^{,rs} W^0 + 2(\rho h\omega^2)^{,r} W^{,s} - D^{,rs}(W^0_{,xxxx} + 2W^0_{,xxyy} + W^0_{,yyyy}) - 2D^{,r}(W^{,s}_{,xxxx} + 2W^{,s}_{,xxyy} + W^{,s}_{,yyyy})\} \operatorname{Cov}(b^r, b^s). \tag{66}$$

Calculating zeroth order terms from the first equation followed sequentially by the first and second order terms, the probabilistic moments of natural frequencies and modes can be obtained. Let us note that the remaining equations of plate vibration can be obtained by an analogous way.

Finally, it should be mentioned that $\varepsilon = 1$ is assumed above to derive the perturbed equations up to the second order in a general form, which makes it possible to obtain the probabilistic moments up to the second central moment (covariance matrix). In further studies, both mathematical and numerical sensitivities of this model with respect to the above assumptions are to be precisely verified. It is known, for instance, that the perturbation approach so defined is effective in computational finite element method-based analyses for standard deviations not higher than about 0·15 of the corresponding expected values [10, 12]. Trying further to generalize the results obtained in equations (52) and (53), it can be observed that when the expected value contains only even order terms, then the covariance matrix is built up with odd order perturbations, so that, for the fourth order perturbation analysis, the following description for the function $W(\mathbf{x})$ expected value holds true

$$E[W] = W^0 + \tfrac{1}{2} W^{,\rho\sigma} S_b^{\xi\sigma} + \tfrac{1}{4!} W^{,\rho\sigma\xi\zeta} S_b^{\rho\sigma\xi\zeta}, \tag{67}$$

where $S_b^{\rho\sigma\xi\zeta}$ denotes the fourth central probabilistic moments of the input parameters. Further, the covariance matrix can be written as

$$S_W^{\alpha\beta} = W^{,\rho} W^{,\sigma} S_b^{\rho\sigma} + W^{,\rho\alpha\beta} W^{,\sigma\xi\zeta} S_b^{\rho\alpha\beta\sigma\xi\zeta}. \tag{68}$$

As it may be recognized from an illustrating example presented in further considerations, most of the third and fourth order derivatives with respect to the random input parameters are equal to 0 for most of the known problems of the elasticity theory and, therefore, the second order approximation is recommended.

4. GENERAL NUMERICAL APPROACH

4.1. DETERMINISTIC MODEL

Let us consider a sufficiently smooth real function $y(\mathbf{x})$ defined discretly by the values y_0, $y_1, y_2, \ldots, y_n$ at uniformly distributed points $x = 0, \delta, 2\delta, \ldots, n\delta$ for $\delta \in \Re_+/\{0\}$ and $n \in N$. The differences at the function values are calculated as [8]

$$(\Delta_1 y)_{x=n\delta} = y_{n+1} - y_n. \tag{69}$$

Then, the first order derivatives of $y(\mathbf{x})$ can be approximated in the form of

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{x=n\delta} \cong \frac{y_{n+1} - y_n}{\delta}. \tag{70}$$

Analogously, starting from the second differences

$$(\Delta_2 y)_{x=n\delta} = (\Delta_1 y)_{x=(n+1)\delta} - (\Delta_1 y)_{x=n\delta} = y_{n+1} - 2y_n + y_{n-1}, \tag{71}$$

we obtain an approximation for the second order derivatives,

$$\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)_{x=n\delta} \cong \frac{(\Delta_2 y)_{x=n\delta}}{\delta^2} = \frac{y_{n+1} - 2y_n + y_{n-1}}{\delta^2}, \quad n \in N. \tag{72}$$

Further, for the sufficiently smooth real function $w = w(x_i)$ and the rectangular planar network defined by the dimension $\delta \in \Re_+/\{0\}$ shown in Figure 1,

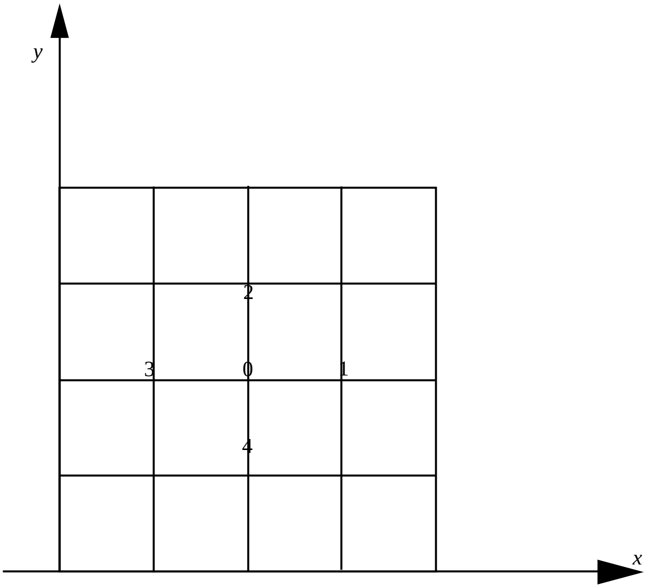

Figure 1. FDM planar rectangular network.

the following approximating equations are obtained:

$$\frac{dw}{dx_1} \cong \frac{w_1 - w_0}{\delta}, \qquad \frac{dw}{dx_2} \cong \frac{w_2 - w_0}{\delta}, \tag{73, 74}$$

$$\frac{d^2w}{dx_1^2} \cong \frac{w_1 - 2w_0 + w_3}{\delta^2}, \qquad \frac{d^2w}{dx_2^2} \cong \frac{w_2 - 2w_0 + w_4}{\delta^2}. \tag{75, 76}$$

Using analogous methodology, the higher order derivatives of function $w(\mathbf{x})$ can also be rewritten. All the equations posed above enable one to solve numerically all these engineering problems which can be formulated in the form of partial differential equations. As it is known, the method can be used in its relaxation version to assure better efficiency of the computations as well as for curved boundaries of the continua, where some correctors for parameter δ should be applied [19, 20].

Extending these equations to up to the fourth order derivatives, the central difference formula for transverse vibrations of a uniform beam can be obtained as

$$W_{i+2} - 4W_{i+1} + (6 - \lambda)W_i - 4W_{i-1} + W_{i-2} = 0, \tag{77}$$

where

$$\lambda = h^4\beta^4. \tag{78}$$

Further, using the planar network presented in Figure 2, the equation of a plate motion can be discretized as

$$\frac{D}{\Delta^4}\left[(W_{i+2,j} + W_{i,j+2} + W_{i-2,j} + W_{i,j-2}) + 2(W_{i+1,j+1} + W_{i-1,j+1} + W_{i-1,j-1} + W_{i+1,j-1}) - 8(W_{i,j+1} + W_{i-1,j} + W_{i,j-1} + W_{i+1,j}) + 20W_{i,j}\right] - \rho h\omega^2 W_{i,j} = 0. \tag{79}$$

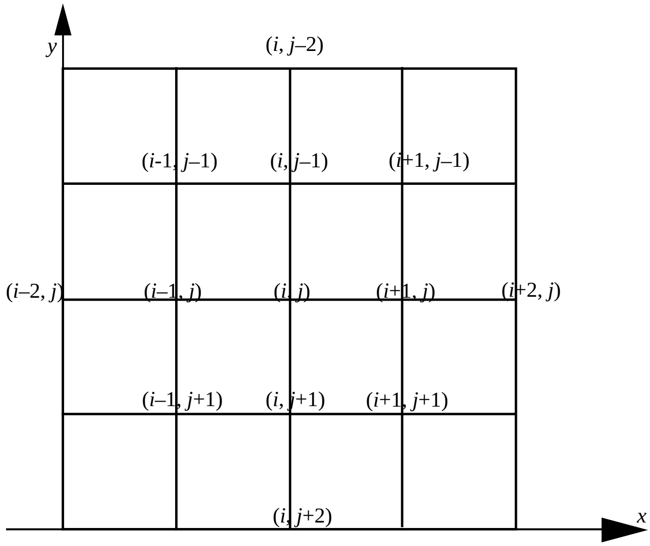

Figure 2. FDM planar network for four-point approximation.

4.2. STOCHASTIC FINITE DIFFERENCE APPROACH

Considering the application of the second order perturbation second probabilistic moment method in the FDM, we start from an approximation of the zeroth, first and second order derivatives of the function $W = W(\mathbf{x})$ being computed with respect to random variable vector components $\{b_r(\mathbf{x};\theta)\}$. For the first partial derivative in x_i direction (assuming the same density of finite difference net in orthogonal directions), the following relations are obtained:

Zeroth order terms:

$$\frac{\partial(W^0)}{\partial x_i} \cong \frac{W_1^0 - W_0^0}{\delta}. \tag{80}$$

First order terms ($r = 1, \ldots, R$):

$$\frac{\partial(W^{,r})}{\partial x_i} = \frac{\partial^2 W}{\partial x_i \partial b_r} \cong \frac{W_1^{,r} - W_0^{,r}}{\delta} = \frac{1}{\delta}\frac{\partial}{\partial b_r}(W_1 - W_0). \tag{81}$$

Second order terms ($r, s = 1, \ldots, R$):

$$\frac{\partial(W^{,rs})}{\partial x_i} = \frac{\partial^2 W}{\partial x_i \partial b_r \partial b_s} \cong \frac{W_1^{,rs} - W_0^{,rs}}{\delta} = \frac{1}{\delta}\frac{\partial}{\partial b_r \partial b_s}(W_1 - W_0). \tag{82}$$

Analogously, the second order derivatives with respect to variable $\mathbf{x}$ can be calculated as

Zeroth order terms:

$$\frac{\partial^2(W^0)}{\partial x_i^2} \cong \frac{W_1^0 - 2W_0^0 + W_3^0}{\delta^2}. \tag{83}$$

First order terms ($r = 1, \ldots, R$):

$$\frac{\partial^2 (W^{,r})}{\partial x_i^2} = \frac{\partial^3 W}{\partial x_i^2 \partial b_r} \cong \frac{W_1^{,r} - W_0^{,r} + W_3^{,r}}{\delta^2} = \frac{1}{\delta^2} \frac{\partial}{\partial b_r} (W_1 - 2W_0 + W_3). \quad (84)$$

Second order terms ($r, s = 1, \ldots, R$):

$$\frac{\partial^2 (W^{,rs})}{\partial x_i^2} = \frac{\partial^2 W}{\partial x_i^2 \partial b_r \partial b_s} \cong \frac{W_1^{,rs} - 2W_0^{,rs} + W_3^{,rs}}{\delta^2} = \frac{1}{\delta^2} \frac{\partial^2}{\partial b_r \partial b_s} (W_1 - 2W_0 + W_3). \quad (85)$$

It should be mentioned that the relevant derivatives of the random variables can be calculated directly starting from the additional probability density functions or numerically, using discrete representations of the characteristic function W. Further, it is observed that the above approximation holds true when the material properties of the structure analyzed are treated as the random variables. In the case of geometrical properties being randomized, the general finite difference scheme is more complicated since $\delta = \delta(\theta)$.

Starting from the above approximations the stochastic second order equations of uniform beam transverse vibrations can be written as follows:

Zeroth order equation:

$$W_{i+2}^0 - 4W_{i+1}^0 + (6 - \lambda^0) W_i^0 - 4W_{i-1}^0 + W_{i-2}^0 = 0. \quad (86)$$

First order equations:

$$W_{i+2}^{,r} - 4W_{i+1}^{,r} + (6 - \lambda^0) W_i^{,r} - 4W_{i-1}^{,r} + W_{i-2}^{,r} = \lambda^{,r} W_i^0. \quad (87)$$

Second order equation:

$$W_{i+2}^{(2)} - 4W_{i+1}^{(2)} + (6 - \lambda^0) W_i^{(2)} - 4W_{i-1}^{(2)} + W_{i-2}^{(2)} = (\lambda^{,rs} W_i^0 + 2\lambda^{,r} W_i^{,s}) \operatorname{Cov}(b^r, b^s). \quad (88)$$

Furthermore, following equations (86–88), up to the second order stochastic free vibrations of a plate can be introduced as

Zeroth order equation:

$$\begin{aligned} D^0 [& (W_{i+2,j}^0 + W_{i,j+2}^0 + W_{i-2,j}^0 + W_{i,j-2}^0) + 2(W_{i+1,j+1}^0 + W_{i-1,j+1}^0 \\ & + W_{i-1,j-1}^0 + W_{i+1,j-1}^0) \\ & - 8(W_{i,j+1}^0 + W_{i-1,j}^0 + W_{i,j-1}^0 + W_{i+1,j}^0) + 20W_{i,j}^0] - (\Delta^4 \rho h \omega^2)^0 W_{i,j}^0 = 0. \end{aligned} \quad (89)$$

First order equations:

$$\begin{aligned} D^0 [& (W_{i+2,j}^{,r} + W_{i,j+2}^{,r} + W_{i-2,j}^{,r} + W_{i,j-2}^{,r}) + 2(W_{i+1,j+1}^{,r} + W_{i-1,j+1}^{,r} \\ & + W_{i-1,j-1}^{,r} + W_{i+1,j-1}^{,r}) \\ & - 8(W_{i,j+1}^{,r} + W_{i-1,j}^{,r} + W_{i,j-1}^{,r} + W_{i+1,j}^{,r}) + 20W_{i,j}^{,r}] = (\Delta^4 \rho h \omega^2)^{,r} W_{i,j}^0 \\ & + (\Delta^4 \rho h \omega^2)^0 W_{i,j}^{,r} - D^{,r} [(W_{i+2,j}^0 + W_{i,j+2}^0 + W_{i-2,j}^0 + W_{i,j-2}^0) \\ & + 2(W_{i+1,j+1}^0 + W_{i-1,j+1}^0 + W_{i-1,j-1}^0 + W_{i+1,j-1}^0) \\ & - 8(W_{i,j+1}^0 + W_{i-1,j}^0 + W_{i,j-1}^0 + W_{i+1,j}^0) + 20W_{i,j}^0]. \end{aligned} \quad (90)$$

Second order equation:

$$D^0[(W^{(2)}_{i+2,j} + W^{(2)}_{i,j+2} + W^{(2)}_{i-2,j} + W^{(2)}_{i,j-2}) + 2(W^{(2)}_{i+1,j+1} + W^{(2)}_{i-1,j+1}$$

$$+ W^{(2)}_{i-1,j-1} + W^{(2)}_{i+1,j-1})$$

$$- 8(W^{(2)}_{i,j+1} + W^{(2)}_{i-1,j} + W^{(2)}_{i,j-1} + W^{(2)}_{i+1,j}) + 20W^{(2)}_{i,j}] = \{(\Delta^4\rho h\omega^2)^{,rs} W^0_{i,j}$$

$$+ 2(\Delta^4\rho h\omega^2)^{,r} W^{,s}_{i,j} - D^{,rs}[(W^0_{i+2,j} + W^0_{i,j+2} + W^0_{i-2,j} + W^0_{i,j-2})$$

$$+ 2(W^0_{i+1,j+1} + W^0_{i-1,j+1} + W^0_{i-1,j-1} + W^0_{i+1,j-1})$$

$$- 8(W^0_{i,j+1} + W^0_{i-1,j} + W^0_{i,j-1} + W^0_{i+1,j}) + 20W^0_{i,j}]$$

$$- 2D^{,r}[(W^{,s}_{i+2,j} + W^{,s}_{i,j+2} + W^{,s}_{i-2,j} + W^{,s}_{i,j-2}) + 2(W^{,s}_{i+1,j+1} + W^{,s}_{i-1,j+1}$$

$$+ W^{,s}_{i-1,j-1} + W^{,s}_{i+1,j-1})$$

$$- 8(W^{,s}_{i,j+1} + W^{,s}_{i-1,j} + W^{,s}_{i,j-1} + W^{,s}_{i+1,j}) + 20W^{,s}_{i,j}]\}\,\mathrm{Cov}(b^r, b^s), \tag{91}$$

where

$$W^{(2)} = W^{,rs} S^{rs}_b. \tag{92}$$

Solving equations (89–92) for the zeroth, first and second orders of random function W, the expected values and covariances of this function are obtained as

$$E[W(x)] = W^0(x) + \tfrac{1}{2} W^{,rs}(x) S^{rs}_b \tag{93}$$

and for the second order moments,

$$\mathrm{Cov}(W(x^{(1)}), W(x^{(2)})) = W^{,r}(x^{(1)}) W^{,s}(x^{(2)}) S^{rs}_b, \tag{94}$$

which completes the general description of the second order perturbation second central probabilistic moment extension of the FDM in case of the vibration of beams and plates.

5. NUMERICAL EXAMPLES

5.1. FIXED–PINNED BEAM-FREE VIBRATIONS

The analytical solution for natural frequencies and modes for the fixed–pinned beam can be obtained as

$$\omega_n^2 = (\beta_n l)^4 \left(\frac{eI}{\rho A l^4}\right), \quad n = 1, 2, \dots, \tag{95}$$

while for the first three modes it is obtained as

$$\omega_n^2 = \begin{bmatrix} 3{\cdot}926602 \\ 7{\cdot}068583 \\ 10{\cdot}210176 \end{bmatrix}^4 \frac{eI}{\rho A l^4}, \quad n = 1, 2, 3. \tag{96}$$

If Young's modulus of a beam is taken as a random variable, then the first two probabilistic moments of natural frequencies square can be calculated using fundamental theorems on random variables as

$$E[\omega_n^2] = \frac{I\beta_n^4}{\rho A} E[e], \quad n = 1, 2, \ldots \tag{97}$$

and

$$\mathrm{Var}(\omega_n^2) = \frac{I^2\beta_n^8}{\rho^2 A^2} \mathrm{Var}(e), \quad n = 1, 2, \ldots \tag{98}$$

Starting from these formulas, the expected values and variances of natural frequencies can be derived as

$$E[\omega] = \sqrt{E^2[\omega]} = \sqrt{E[\omega^2] - \mathrm{Var}(\omega)} \tag{99}$$

and

$$\mathrm{Var}(\omega^2) = 2\,\mathrm{Var}(\omega)(\mathrm{Var}(\omega) + 2E^2[\omega]). \tag{100}$$

Using stochastic second order perturbation approach, the same probabilistic characteristics can be calculated as

$$E[\omega_n^2] = (\omega_n^2)^0 + \tfrac{1}{2}(\omega_n^2)^{,rs} \mathrm{Cov}(b^r, b^s) = \frac{I\beta_n^4}{\rho A} E[e], \quad n = 1, 2, \ldots, \tag{101}$$

as well as

$$\mathrm{Var}(\omega_n^2) = (\omega_n^2)^{,r}(\omega_n^2)^{,s} \mathrm{Cov}(b^r, b^s) = \frac{I^2\beta_n^8}{\rho^2 A^2} \mathrm{Var}(e), \quad n = 1, 2, \ldots. \tag{102}$$

As it can be observed, the analytical solution for probabilistic moments calculated from the definition and the second order perturbation approach are exactly the same and that is why there is no need to apply the higher order perturbation approach; it generally holds true for all output linear functions of input random variables. For the other type of relations between output frequencies and system random parameters, the expected values obtained from the second order perturbation approach are higher than those derived from the definition.

Analogous calculations are done using stochastic finite difference method (SFDM). Let us consider the discretization of a fixed–pinned beam presented in Figure 3.

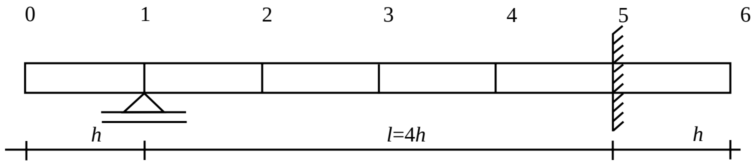

Figure 3. FDM discretization of a fixed–pinned beam.

By introducing fictitious nodes 0 and 6, respectively, the expected values and variances are obtained by the use of the perturbation technique assuming a rectangular cross-section $b \times h$ of the beam as follows:

Expected values:

$$E[\omega_n^2] = (\tilde{\beta}_n)^4 \frac{h^2}{12\rho} E[e]; \tag{103}$$

Variances:

$$\mathrm{Var}(\omega_n^2) = (\tilde{\beta}_n)^8 \frac{h^4}{144\rho^2} \mathrm{Var}(e); \tag{104}$$

where $\tilde{\beta}_n = \{3{\cdot}3788, 8{\cdot}9732, 14{\cdot}0024\}$. It should be underlined here that the main source of the differences between analytical and FDM results from the partition of the beam into only four intervals only.

5.2. SIMPLY SUPPORTED SQUARE PLATE

Let us consider vibrations of the square plate of thickness h, characterized by Young's modulus e, the Poisson ratio ν and mass density ρ. Using the FDM discretization for the plate shown below (see Figure 4), the following finite difference equation is obtained:

$$\left(18 - \Delta^4 \frac{\rho h \omega^2}{D}\right) W_{3,3} - 8W_{4,3} - 8W_{3,4} + 2W_{4,4} = 0, \tag{105}$$

where $\Delta = a/3$. The natural frequencies are obtained as

$$\omega_n = \frac{\lambda_n}{a^2}\sqrt{\frac{D}{\rho h}} = \frac{\lambda_n}{a^2}\sqrt{\frac{eh^2}{12\rho(1-\nu^2)}}, \tag{106}$$

where the coefficients λ_n are obtained as $\{19{\cdot}7392; 49{\cdot}35; 49{\cdot}35; 789569\}$.

Analyzing, as in the previous example, the square of natural frequencies and considering plate thickness as a random variable, the expected values are obtained as

$$E[\omega_n^2] = \frac{\lambda_n^2}{a^2}\frac{e}{12\rho(1-\nu^2)} E[h^2] = \frac{\lambda_n^2}{a^2}\frac{e}{12\rho(1-\nu^2)} (E^2[h] + \mathrm{Var}(h)) \tag{107}$$

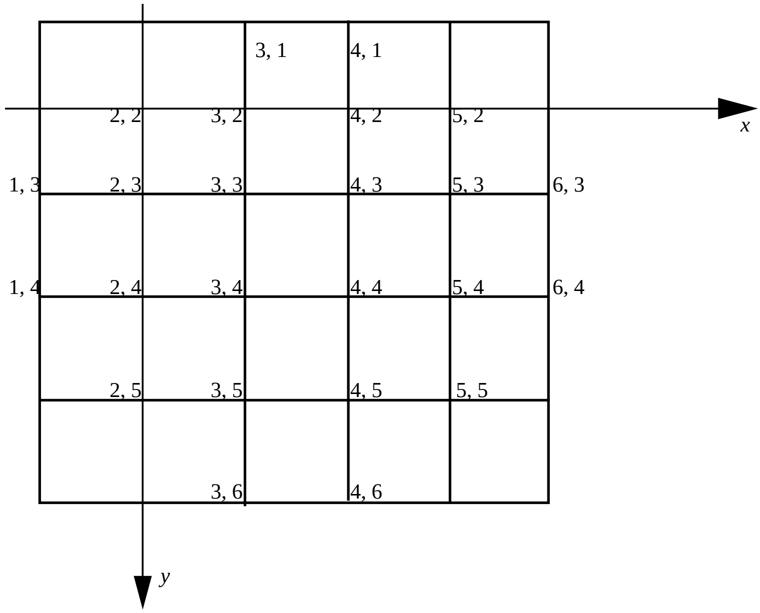

Figure 4. FDM discretization for a square plate.

for direct derivation using the definition. On the other hand, using the second order perturbation technique, the result is exactly the same for "h" being a Gaussian random variable:

$$E[\omega_n^2] = (\omega_n^2)^0 + \tfrac{1}{2}(\omega_n^2)^{,hh}\,\mathrm{Var}(h) = \frac{\lambda_n^2}{a^2}\frac{e}{12\rho(1-\nu^2)}E^2[h], \tag{108}$$

where $\lambda = \{18{\cdot}0;\ 36{\cdot}0;\ 36{\cdot}0;\ 54{\cdot}0\}$. Further, the variances are obtained for the analytical derivation,

$$\mathrm{Var}(\omega_n^2) = \frac{\lambda_n^4}{a^4}\frac{e^2}{144\rho^2(1-\nu^2)^2}\mathrm{Var}(h^2), \tag{109}$$

which, using the relation characteristic for Gaussian variables, gives

$$\mathrm{Var}(h^2) = \mathrm{Var}^2(h) + 2\,\mathrm{Var}(h)E^2[h] \tag{110}$$

can be rewritten as

$$\mathrm{Var}(\omega_n^2) = \frac{\lambda_n^4}{a^4}\frac{e^2}{144\rho^2(1-\nu^2)^2}(\mathrm{Var}^2(h) + 2\,\mathrm{Var}(h)E^2[h]). \tag{111}$$

Using the SFDM procedure, it can be calculated that

$$\mathrm{Var}(\omega_n^2) = (\omega_n^2)^{,h}(\omega_n^2)^{,h}\,\mathrm{Var}(h) = \frac{\lambda_n^4}{a^4}\frac{4e^2E^2[h]}{144\rho^2(1-\nu^2)^2}\mathrm{Var}(h). \tag{112}$$

It should be noted that if output frequencies are linear functions of some parameter, then the corresponding probabilistic moments obtained by analytical and perturbation technique are equal to each other, respectively.

6. CONCLUDING REMARKS

The second order perturbation second central probabilistic moment extension of the classical finite difference methods (FDM) presented above can be successfully applied in the numerical analysis of the typical stochastic partial differential equation systems that can be solved in deterministic case. Further, it can be observed that the space discretization formulation for the FDM obtained above may be easily reformulated for the time-domain discretization. For this purpose, the spatial correlations for co-ordinates $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are to be replaced with time moments $t^{(1)}$ and $t^{(2)}$, respectively. Such a formulation may be useful in a numerical simulation where the finite element method is improved for space discretization, while the FDM is applied for time incrementation procedure and, on the other hand, in the wavelet-optimized finite difference method (WOFDM) [21] which has become more and more popular; see the recently enormously growing engineering applications field of the wavelet analysis. Next, it can be observed that the stochastic finite difference method (SFDM) worked out may be applied for regular as well as irregular grids and, on the other hand, can be relatively easily extended to cases where the relaxation method should be used.

Having closed-form equations for deterministic output, probabilistic moments can be computed using the second order perturbation technique implemented in any symbolic computations mathematical package, as it is demonstrated above in the example of the program MAPLE [22]. It should be underlined that starting from the closed-form equations for the probabilistic moments of output function, the stochastic sensitivity equations for different probabilistic moments of input structural parameters can be derived. It enables engineers to verify if the structural response is more sensitive to expected values of the given structural parameter and/or to its standard deviations.

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